Vector fields on differentiable schemes and derivations on differentiable rings (Singularity theory of differential maps and its applications)

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数理解析研究所講究録
第 1948 巻 2015 年 57-64

Vector fields on differentiable schemes and derivations on differentiable rings

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1 Introduction

This is a survey paper which was presented by the author in the RIMS at Kyoto University. This paper contains several recent results which obtained by the author. Let us mention on the motivations of our study related to the theory of manifolds and that of C^{∞}-rings.

Let M, N be C^{∞}-manifolds and $f : N \rightarrow M$ a C^{∞}-map. Write an $\mathbb{R}$-algebra $C^{\infty}(M)$ as a set of C^{∞}-functions on M, and a homomorphism $f^{*} : C^{\infty}(M) \rightarrow C^{\infty}(N)$ defined as $f^{*}(h) := h \circ f$.

We can regard a vector field $V : N \rightarrow TM$ along f as an **$\mathbb{R}$-derivation** $V : C^{\infty}(M) \rightarrow C^{\infty}(N)$ by f^{*} i.e. V is an $\mathbb{R}$-linear map such that

$$V(h_1h_2) = f^{*}(h_1) \cdot V(h_2) + f^{*}(h_2) \cdot V(h_1) \text{ for any } h_1, h_2 \in C^{\infty}(M).$$

Note that in this case, V turns to be a **C^{∞}-derivation**, i.e. V satisfies that

$$V\big(g \circ (h_1, \ldots, h_l)\big) = \sum_{i=1}^{l} f^{*}\Big(\frac{\partial g}{\partial x_i} \circ (h_1, \ldots, h_l)\Big) \cdot V(h_i)$$
$$\text{for any } l \in \mathbb{N},\ h_1, \ldots, h_l \in C^{\infty}(M), \text{and } g \in C^{\infty}(\mathbb{R}^l).$$

$C^{\infty}(M)$ is a kind of "**C^{∞}-ring**" with the property: for any $l \in \mathbb{N}$ and $g \in C^{\infty}(\mathbb{R}^l)$, there exists an operation $\Phi_g : C^{\infty}(M)^l \rightarrow C^{\infty}(M)$ defined as $\Phi_g(h_1, \ldots, h_l) := g \circ (h_1, \ldots, h_l)$. For C^{∞}-rings $\mathfrak{C}, \mathfrak{D}$ and a homomorphism $\phi : \mathfrak{C} \rightarrow \mathfrak{D}$ of C^{∞}-rings, when does an $\mathbb{R}$-derivation $v : \mathfrak{C} \rightarrow \mathfrak{D}$ over ϕ become a C^{∞}-derivation?

C^{∞}-ringed spaces are sheaves with C^{∞}-rings. There exists a functor Spec : $\mathbf{C^{\infty}Rings}^{\mathrm{op}} \rightarrow \mathbf{LC^{\infty}RS}$ such that C^{∞}-manifolds are regarded as "**C^{∞}-schemes**" $M = Spec\big(C^{\infty}(M)\big)$. We can regard a C^{∞}-manifold M as a "space" associated with $C^{\infty}(M)$ and a vector field over M as a derivation $C^{\infty}(M) \rightarrow C^{\infty}(M)$ by the functor *Spec*.

Then, what should we regard as a vector field on C^{∞}-scheme? To define and study of singular points and vector fields on C^{∞}-schemes, we study <u>properties of derivations $V : \mathfrak{C} \rightarrow \mathfrak{C}$ of C^{∞}-rings.</u>

At §2, we refer to C^{∞}-rings. First, we illustrate the definition of C^{∞}-rings and the examples of C^{∞}-rings, like the set $C^{\infty}(M)$ of the smooth functions on a smooth manifold M. Second, we illustrate the definition of $\mathbb{R}$-derivations, C^{∞}-derivations on C^{∞}-rings, cotangent module and the example of derivations on $C^{\infty}(M)$. Moreover, we define k-jet projection on C^{∞}-rings and k-jet determined $C\ \infty$-rings to find out the relation of derivations on C^{∞}-rings.

At §3, we refer to the theorem of the condition that any $\mathbb{R}$-derivation becomes C^{∞}-derivaiton. First, we illustrate the necessary and sufficient condition that any $\mathbb{R}$-derivation becomes C^{∞}-derivaiton. To illustrate the condition, we use a free C^{∞}-module generated by $d(c)$ and its two submodules related to derivations. Second, we illustrate the theorem of this survey. Moreover, we illustrate the example of derivations on k-jet determined C^{∞}-rings and the example of C^{∞}-rings which have $\mathbb{R}$-derivations which is not C^{∞}-derivation.

2 Differentiable rings and their derivations

First, we refer to the definition of C^∞-rings from [4] and [2].

Definition 2.1 (E. J. Dubuc, c.f. D. Joyce) *1. A* **C^∞-ring** *(***differentiable ring***) is a set $\mathfrak{C}$ which satisfies that: for any $l \in \{0\} \cup \mathbb{N}$ and any C^∞-map $f : \mathbb{R}^l \to \mathbb{R}$, there exists an operation $\Phi_f : \mathfrak{C}^l \to \mathfrak{C}$ such that*

- *for any $k \in \{0\} \cup \mathbb{N}$, any C^∞-maps $g : \mathbb{R}^k \to \mathbb{R}$ and $f_i : \mathbb{R}^l \to \mathbb{R} (i = 1, \cdots, k)$,*

$$\Phi_g(\Phi_{f_1}(c_1, \ldots, c_l), \ldots, \Phi_{f_k}(c_1, \ldots, c_l)) = \Phi_{g\circ(f_1,\ldots,f_k)}(c_1, \ldots, c_l)$$
for any $c_1, \cdots, c_l \in \mathfrak{C}$ and,

- *for all projections $\pi_i(x_1, \ldots, x_l) = x_i (i = 1, \cdots, l)$,*

$$\Phi_{\pi_i}(c_1, \ldots, c_l) = c_i \text{ for any } c_1, \cdots, c_l \in \mathfrak{C}.$$

2. Let $\mathfrak{C}$ and $\mathfrak{D}$ be C^∞-rings. A morphism between C^∞-rings is a map $\phi : \mathfrak{C} \to \mathfrak{D}$ such that

$$\phi(\Phi_f(c_1, \ldots, c_n)) = \Psi_f(\phi(c_1), \ldots, \phi(c_n))$$
for any $n \in \mathbb{N}, f \in C^\infty(\mathbb{R}^n)$ and $c_1, \ldots, c_n \in \mathfrak{C}$.

3. We will write ***C^∞Rings*** *for the category of C^∞-rings.*

Any C^∞-ring $\mathfrak{C}$ has a structure of the commutative $\mathbb{R}$-algebra. Define addtion on $\mathfrak{C}$ by $c + c' := \Phi_{(x,y)\mapsto x+y}(c, c')$. Define multiplication on $\mathfrak{C}$ by $c \cdot c' := \Phi_{(x,y)\mapsto xy}(c, c')$. Define scalar multiplication by $\lambda \in \mathbb{R}$ by $\lambda c := \Phi_{x\mapsto\lambda x}(c)$. Define elements 0 and 1 in $\mathfrak{C}$ by $0_{\mathfrak{C}} := \Phi_{\emptyset\mapsto 0}(\emptyset)$ and $1_{\mathfrak{C}} := \Phi_{\emptyset\mapsto 1}(\emptyset)$.

Example 2.1 *1. Suppose that M is a C^∞-manifold.*

(a) The set $C^\infty(M)$ has a structure of C^∞-ring by $(c_1, \ldots, c_n) \mapsto f \circ (c_1, \ldots, c_n)$.

(b) Let $I \subset C^\infty(M)$ be an ideal of an $\mathbb{R}$-algebra. We can define a quotient $\mathbb{R}$-algebra $C^\infty(M)/I$. For any natural number $l \in \mathbb{N}$ and a C^∞-map $f \in C^\infty(\mathbb{R}^l)$, $f(x_1 + y_1, \ldots, x_l + y_l) - f(x_1, \ldots, x_l) = \sum_{i=1}^{l} y_i g_i(x, y)$ by Hadamard's lemma. Then $f \circ (c_1 + i_1, \ldots, c_n + i_n) - f \circ (c_1, \ldots, c_n) = \sum_{k=1}^{n} i_k \cdot g_k \circ (c_1, \ldots, c_n, i_1, \ldots, i_n)$ for any $c_1, \ldots, c_n \in \mathfrak{C}$ and $i_1, \ldots, i_n \in I$. Therefore the $\mathbb{R}$-algebra $C^\infty(M)/I$ has a structure of C^∞-ring.

(c) The set $C^\infty_p(M)/{m_p}^{k+1}$ of k-jet functions on a point $p \in M$ has a structure of C^∞-ring.

2. The set of real numbers $\mathbb{R}$ has a structure of C^∞-ring by $(r_1, \ldots, r_n) \mapsto f(r_1, \ldots, r_n)$.

Second, we refer to the definition of two derivations on C^∞-rings as followings.

Definition 2.2 (R. Hartshorne, D. Joyce) *Let $\mathfrak{C}$ be a C^∞-ring and $\mathfrak{M}$ be a $\mathfrak{C}$-module.*

1. An **$\mathbb{R}$-derivation** *is an $\mathbb{R}$-linear map $d : \mathfrak{C} \to \mathfrak{M}$ such that*

$$d(c_1c_2) = c_2 \cdot d(c_1) + c_1 \cdot d(c_2) \text{ for any } c_1, c_2 \in \mathfrak{C}.$$

2. A **C^∞-derivation** *is an $\mathbb{R}$-linear map $d : \mathfrak{C} \to \mathfrak{M}$ such that*

$$d(\Phi_f(c_1, \ldots, c_n)) = \sum_{i=1}^{n} (\Phi_{\frac{\partial f}{\partial x_i}}(c_1, \ldots, c_n)) \cdot d(c_i)$$
for any $n \in \mathbb{N}, f \in C^\infty(\mathbb{R}^n)$ and $c_1, \ldots, c_n \in \mathfrak{C}$.

By definition, we have that any C^∞-derivation is an $\mathbb{R}$-derivation.

Definition 2.3 (R. Hartshorne, D. Joyce) *Let $\mathfrak{C}$ be a C^∞-ring , $\mathfrak{M}$ $\mathfrak{C}$-module and $d : \mathfrak{C} \to \mathfrak{M}$ $\mathbb{R}$-derivation(resp. C^∞-derivation). We call a pair $(\mathfrak{M}, d)$ an* **$\mathbb{R}$-cotangent module** *(resp. a* **C^∞-cotangent module***) for $\mathfrak{C}$ if $(\mathfrak{M}, d)$ satisfies that*

for any $\mathbb{R}$-derivation (resp. C^∞-derivation) $d' : \mathfrak{C} \to \mathfrak{M}'$
there exists a unique morphism $\phi : M \to \mathfrak{M}'$ of $\mathfrak{C}$-modules such that $\phi \circ d = d'$.

We write $(\Omega_{\mathfrak{C},\mathbb{R}}, d_{\mathfrak{C},\mathbb{R}})$ (resp. $(\Omega_{\mathfrak{C},C^\infty}, d_{\mathfrak{C},C^\infty})$) for the $\mathbb{R}$-cotangent module (resp. the C^∞-cotangent module) for $\mathfrak{C}$.

For the uniqueness of cotangent modules, there exists a unique morphism $\Omega_\phi : \Omega_{\mathfrak{C}} \to \Omega_{\mathfrak{D}}$ of $\mathfrak{C}$-modules with a following property:

$$d_{\mathfrak{D}} \circ \phi \equiv \Omega_\phi \circ d_{\mathfrak{C}} : \mathfrak{C} \longrightarrow \Omega_{\mathfrak{D}}.$$

Example 2.2 *For a C^∞-manifold M and $C^\infty(M)$, $\Omega_{C^\infty(M),\mathbb{R}}$ and $\Omega_{C^\infty(M),C^\infty}$ are isomorphic to the set $\Gamma(T^*M)$ of C^∞-sections to the cotangent bundle T^*M on M.*

Example 2.3 *Let M be a C^∞-manifold.*

1. *For any $f \in C^\infty(M)$, define a smooth function $df \in \Gamma(T^*M)$ as $df(v) := v(f)$ for any $x \in M$ and $v \in T_xM$. Define $\mathbb{R}$-mapping $d : C^\infty(M) \to \Gamma(T^*M)$ as $d(f) := df$. This $\mathbb{R}$-mapping d is the C^∞-derivation.*

2. *Let $V : M \to TM$ be a C^∞-vector field of M. For any $f \in C^\infty(M)$, define a smooth function $V(f) \in C^\infty(M)$ as $\big(V(f)\big)(x) := V_x(f)$. We can regard $V : C^\infty(M) \to C^\infty(M)$ as the $\mathbb{R}$-derivation. This $\mathbb{R}$-derivation is also C^∞-derivation.*

We define k-jet projections on C^∞-rings and k-jet determined C^∞-rings. To define them, first, we illustrate the definition of $\mathbb{R}$-points of C^∞-rings and localizations of C^∞-rings.

Definition 2.4 (D. Joyce) *Let $\mathfrak{C}$ be a C^∞-ring.*

1. *An* **$\mathbb{R}$-point of $\mathfrak{C}$** *is a homomorphism $p : \mathfrak{C} \to \mathbb{R}$ of C^∞-rings.*
   The set of $\mathbb{R}$-points $p : \mathfrak{C} \to \mathbb{R}$ is a base space of the C^∞-scheme $Spec\mathfrak{C}$.

2. *Suppose that the morphism $p : \mathfrak{C} \to \mathbb{R}$ is an $\mathbb{R}$-point. The* **localization** *$\mathfrak{C}_p$ always exists with the unique maximal ideal $m_p \subset \mathfrak{C}_p(\mathfrak{C}_p/m_p \cong \mathbb{R})$, i.e. there exists a unique C^∞-ring $\mathfrak{C}_p$ and its unique ideal m_p which satisfy*
   - *there exists a morphism $\pi_p : \mathfrak{C} \to \mathfrak{C}_p$ such that $\pi_p(s)$ is invertible for any $s \in p^{-1}(\mathbb{R}\backslash\{0\})$,*
   - *for any morphism $\pi'_p : \mathfrak{C} \to \mathfrak{C}'$ such that $\pi_p(s)$ is invertible for any $s \in p^{-1}(\mathbb{R}\backslash\{0\})$, there exists a unique $\phi : \mathfrak{C}_p \to \mathfrak{C}'$ with $\pi'_p \equiv \phi \circ \pi_p$ and*
   - *$\mathfrak{C}_p/m_p$ is isomorphic to $\mathbb{R}$.*

3. *For any nonnegative number $k \in \{0\} \cup \mathbb{N} \cup \{\infty\}$, define natural projections as*

$$j_p^k : \mathfrak{C} \to \mathfrak{C}_p/{m_p}^{k+1},$$
$$j^k := (j_p^k)_{p:\mathfrak{C}\to\mathbb{R}} : \mathfrak{C} \to \prod_{p:\mathfrak{C}\to\mathbb{R}} \mathfrak{C}_p/m_p^{k+1}.$$

(If $k = \infty$, we write ${m_p}^{k+1} := {m_p}^\infty := \cap_{k\in\mathbb{N}}{m_p}^k$).

From this example about C^∞-manifolds, localizations of C^∞-rings by $\mathbb{R}$-points are the generalization of germ of smooth functions on C^∞-manifolds.

Example 2.4 *Let M be a C^∞-manifold and $p \in M$. For the $\mathbb{R}$-point $e_p : C^\infty(M) \to \mathbb{R}$ as $e_p(f) := f(p)$, a localization $\left(C^\infty(M)\right)_{e_p}$ is isomorphic to the set $C_p^\infty(M)$ of germs of C^∞-functions at p. Its unique maximal ideal is $m_{e_p} = \{[f,U]_p \in C_p^\infty(M) | f(p) = 0\}$.*

We define k-jet determined C^∞-rings and its properties for direct product.

Definition 2.5 (1,I. Moerdijk and G.E. Reyes, 2,3,Yamashita) *Let $\mathfrak{C}$ be a C^∞-ring.*
Let $k \in \{0\} \cup \mathbb{N} \cup \{\infty\}$. $\mathfrak{C}$ is **k-jet determined** *if $j^k : \mathfrak{C} \to \prod_{p:\mathfrak{C}\to\mathbb{R}} \mathfrak{C}_p/m_p^{k+1}$ is injective. Espencially, 0-jet determined C^∞-rings are called* **point determined C^∞-rings**.

Example 2.5 *Suppose that M is a C^∞-manifiold.*

1. *$C^\infty(M)$ is a point determined C^∞-ring.*
2. *$C_p^\infty(M)/{m_p}^{k+1}$ is not a point determined C^∞-ring, but a k-jet determined C^∞-ring.*

For two C^∞-rings $\mathfrak{C}$ and $\mathfrak{D}$ with operations $\Phi_f : \mathfrak{C}^n \to \mathfrak{C}$ and $\Psi_f : \mathfrak{D}^n \to \mathfrak{D}$ for $f \in C^\infty(\mathbb{R}^n)$, we can define a direct product $\mathfrak{C} \times \mathfrak{D}$. This product has a structure of C^∞-ring by $\Xi_f : (\mathfrak{C} \times \mathfrak{D})^n \to \mathfrak{C} \times \mathfrak{D}$ defined as

$$\Xi_f\big((c_1,d_1),\dots,(c_n,d_n)\big) := \big(\Phi_f(c_1,\dots,c_n), \Psi_f(d_1,\dots,d_n)\big).$$

Any $\mathbb{R}$-point $e : \mathfrak{C} \times \mathfrak{D} \to \mathbb{R}$, there exists a unique $\mathbb{R}$-point $e' : \mathfrak{C} \to \mathbb{R}$ or $e' : \mathfrak{D} \to \mathbb{R}$ such that $e' \circ \pi_{\mathfrak{C}} = e$ or $e' \circ \pi_{\mathfrak{D}} = e$. Hence, $(\mathfrak{C} \times \mathfrak{D})_e$ is isomorphic to $\mathfrak{C}_{e'}$ or $\mathfrak{D}_{e'}$ and we have a following property for direct product of k-jet determined C^∞-rings.

Proposition 2.1 (Yamashita) *Let $\mathfrak{C}$ and $\mathfrak{D}$ be k,l-jet determined C^∞-rings and $k' := \max(k,l)$.*
The direct product $\mathfrak{C} \times \mathfrak{D}$ is a k'-jet determined C^∞-ring.

Example 2.6 *Let M and M' be m-dimensional C^∞-manifolds. Write $M \sqcup M'$ as a disjoint union of C^∞-manifolds M and M'. $C^\infty(M)$ and $C^\infty(M')$ are point determined C^∞-rings.*
Furthermore, $C^\infty(M) \times C^\infty(M') \doteq C^\infty(M \sqcup M')$ is a point determined C^∞-ring, too.

Let $\mathfrak{C}$ be a C^∞-ring. For $k,l \in \{0\} \cup \mathbb{N} \cup \{\infty\} (k \le l)$, we have the homomorphism
$j^{k,l} : \prod_{p:\mathfrak{C}\to\mathbb{R}} \mathfrak{C}_p/{m_p}^{l+1} \to \prod_{p:\mathfrak{C}\to\mathbb{R}} \mathfrak{C}_p/{m_p}^{k+1}$ such that $j_p^k = j_p^{k,l} \circ j_p^l$ for any $p : \mathfrak{C} \to \mathbb{R}$.

$$\begin{array}{ccc} \mathfrak{C} & \xrightarrow{j^l} & \prod_{p:\mathfrak{C}\to\mathbb{R}} \mathfrak{C}_p/{m_p}^{l+1}. \\ & j^k \searrow & \downarrow j^{k,l} \\ & & \prod_{p:\mathfrak{C}\to\mathbb{R}} \mathfrak{C}_p/{m_p}^{k+1} \end{array}$$

If j^k is injective, j^l is also injective. Therefore, we have the following proposition.

Proposition 2.2 (Yamashita) *Let $\mathfrak{C}$ be a C^∞-ring and $k,l \in \{0\} \cup \mathbb{N} \cup \{\infty\} (k \le l)$.*
If $\mathfrak{C}$ is a k-jet determined C^∞-ring, then $\mathfrak{C}$ is also l-jet determined.

3 Algebraic viewpoints

From the definitions of $\mathbb{R}$-cotangent modules and C^∞-cotangent modules, we have a following property.

Proposition 3.1 (Yamashita) *Let $\mathfrak{C}$ be a C^∞-ring and $\mathfrak{F}_{\mathfrak{C}}$ a free $\mathfrak{C}$-module generated by $d(c) (c \in \mathfrak{C})$.*
Define elements

$$\begin{aligned} s(c,c') &:= d(c+c') - d(c) - d(c') \\ p(c,\lambda) &:= d(\lambda c) - \lambda d(c) \\ R(c,c') &:= d(c \cdot c') - c d(c') - c' d(c) \\ C^\infty(c_1,\dots,c_n;f) &:= d\big(\Phi_f(c_1,\dots,c_n)\big) - \sum_{i=1}^n \Phi_{\frac{\partial f}{\partial x_i}}(c_1,\dots,c_n) d(c_i) \end{aligned}$$

for any $n \in \mathbb{N}$, $f \in C^\infty(\mathbb{R}^n)$ *and* $c, c', c_1, \ldots, c_n \in \mathfrak{C}$.

Define $\mathfrak{M}_{\mathfrak{C},\mathbb{R}}$ *as a* $\mathfrak{C}$*-submodule of* $\mathfrak{F}_{\mathfrak{C}}$ *which is generated by* $s(c,c'), p(c,\lambda), R(c,c')$, *and* $\mathfrak{M}_{\mathfrak{C},C^\infty}$ *as a* $\mathfrak{C}$*-submodule of* $\mathfrak{F}_{\mathfrak{C}}$ *which is generated by* $s(c,c'), p(c,\lambda), C^\infty(c_1,\ldots,c_n;f)$.

$\mathfrak{M}_{\mathfrak{C},\mathbb{R}} = \mathfrak{M}_{\mathfrak{C},C^\infty}$ *if and only if any* $\mathbb{R}$*-derivation* $d : \mathfrak{C} \to \mathfrak{M}$ *become a* C^∞*-derivation.*

Suppose that a C^∞-ring $\mathfrak{C}$ is a local C^∞-ring with a unique maximal ideal m such that $m^{k+1} = 0$. Take any C^∞-function $f \in C^\infty(\mathbb{R}^n)$. By Hadamard's lemma, there exists smooth functions $G_{\alpha,p}, G^i_{\alpha,p} : \mathbb{R}^n \to \mathbb{R}(|\alpha| = k+2, i = 1,\ldots,n)$ such that

$$f(x) = f(p) + \sum_{1\le|\alpha|\le k+1} \frac{(x-p)^\alpha}{\alpha!}\frac{\partial^\alpha f}{\partial x^\alpha}(p) + \sum_{|\alpha|=k+2}(x-p)^\alpha G_{\alpha,p}(x),$$

$$\frac{\partial f}{\partial x_i}(x) = \sum_{0\le|\alpha|\le k} \frac{(x-p)^\alpha}{\alpha!}\frac{\partial^{\alpha+e_i} f}{\partial x^{\alpha+e_i}}(p) + \sum_{|\alpha|=k+1}(x-p)^\alpha G^i_{\alpha,p}(x).$$

For any $\mathbb{R}$-derivation $d : \mathfrak{C} \to \mathfrak{M}$, we have $\sum_{i=1}^n \phi(\Phi_{\frac{\partial f}{\partial x_i}}(c_1,\ldots,c_n))d(c_i) = d(\Phi_f(c_1,\ldots,c_n))$ from $m^{k+1} = 0$ for $k \in \{0\} \cup \mathbb{N}$ or $m^\infty = \cap_{k=0}^\infty m^{k+1} = 0$. Therefore, we have a following theorem from k-jet determined C^∞-rings for $k \in \{0\} \cup \mathbb{N} \cup \{\infty\}$.

Theorem 3.1 (Yamashita) *Let* $\mathfrak{C}, \mathfrak{D}$ *be* C^∞*-rings,* $\phi : \mathfrak{C} \to \mathfrak{D}$ *a homomorphism of* C^∞*-rings and* $k \in \mathbb{N} \cup \{\infty\}$. *Suppose that* $\mathfrak{D}$ *is point determined or k-jet determined.*
Then any $\mathbb{R}$*-derivation* $V : \mathfrak{C} \to \mathfrak{D}$ *over* ϕ *is a* C^∞*-derivation.*

Example 3.1 *1. Let* V *be an* $\mathbb{R}$*-derivation* $V : C^\infty(M) \to C^\infty(N)$ *over the pull-back* $f^* : C^\infty(M) \to C^\infty(N)$. $C^\infty(N)$ *is a point determined* C^∞*-ring. From the previous theorem, this* $\mathbb{R}$*-derivation is a* C^∞*-derivation.*

2. $C^\infty(\mathbb{R})/\langle x^{k+1}\rangle_{C^\infty(\mathbb{R})}$ *is not point determined but k-jet determined* C^∞*-ring.*
Any $\mathbb{R}$*-derivation* $V : C^\infty(\mathbb{R})/\langle x^{k+1}\rangle_{C^\infty(\mathbb{R})} \to C^\infty(\mathbb{R})/\langle x^{k+1}\rangle_{C^\infty(\mathbb{R})}$ *is* C^∞*-derivation such that*

$$V(f(x) + \langle x^{k+1}\rangle) = \frac{\partial f}{\partial x}(x)v(x) + \langle x^{k+1}\rangle$$
$$\text{by } v(x) + \langle x^{k+1}\rangle := V(x + \langle x^{k+1}\rangle).$$

For the previous example , we have a following corollary by generalizing $C^\infty(\mathbb{R})/\langle x^{k+1}\rangle_{C^\infty(\mathbb{R})}$.

Corollary 3.1 (Yamashita) *Let* $\mathfrak{C}$ *be a k-jet determined* C^∞*-ring with the form* $C^\infty(\mathbb{R}^n)/I$.
For any $\mathbb{R}$*-derivation* $V : \mathfrak{C} \to \mathfrak{C}$, V *is a* C^∞*-derivation.*
Moreover, there exists n smooth functions $a_1,\ldots,a_n(x) \in C^\infty(\mathbb{R}^n)$ *which satisfy*

$$V(f(x) + I) = \sum_{i=1}^n a_i(x)\frac{\partial f}{\partial x_i}(x) + I \text{ for any } f(x) + I \in C^\infty(\mathbb{R}^n)/I.$$

From Remark 5.12 in [2], we have a counterexample of C^∞-rings such that any $\mathbb{R}$-derivations become C^∞-derivation.

Remark 3.1 *For the* C^∞*-ring* $C^\infty(\mathbb{R}^n)$, $\Omega_{C^\infty(\mathbb{R}^n),\mathbb{R}}$ *is generally much larger than* $\Omega_{C^\infty(\mathbb{R}^n),C^\infty}$, *so that* $\Omega_{C^\infty(\mathbb{R}^n),\mathbb{R}}$ *is not a finitely generated* $C^\infty(\mathbb{R}^n)$*-module for* $n > 0$.

4 Applications

4.1 Applications to C^∞-vector field along C^∞-curve

Let $\mathfrak{C}$ be a C^∞-ring and $\phi : \mathfrak{C} \to C^\infty(\mathbb{R})$ a homomorphism of C^∞-rings. This homomorphism is regarded as a C^∞-curve $\mathbb{R} \to Spec\mathfrak{C}$.

Suppose that $V : \mathfrak{C} \to C^\infty(\mathbb{R})$ is an $\mathbb{R}$-derivation over ϕ. For the previous theorem, this derivation V is a C^∞-derivation. Furthermore, C^∞-derivation V is regarded as a tangent vector at $Spec\mathfrak{C}$.

For any element $c' \in \mathfrak{C}$, we can define a homomorphism $\psi : \mathfrak{C} \to \mathfrak{C}$ of C^∞-rings as $\psi(c) := \Phi_{\phi(c)}(c')$, and a C^∞-derivation $V' : \mathfrak{C} \to \mathfrak{C}$ over ψ as $V'(c) := \Phi_{V(c)}(c')$.

4.2 Applications to C^∞-vector field along C^∞-map

Let $\mathfrak{C}$ be a C^∞-ring, M a C^∞-manifold and $\phi : \mathfrak{C} \to C^\infty(M)$ a homomorphism of C^∞-rings.

Suppose that $V : \mathfrak{C} \to C^\infty(M)$ is an $\mathbb{R}$-derivation by ϕ. For the previous theorem, this derivation V is a C^∞-derivation.

Therefore, we can define a vector field $V : M \to Spec\mathfrak{C}$ over $Spec\phi : M \to Spec\mathfrak{C}$ as the image of derivation $\mathfrak{C} \to C^\infty(M)$ by the functor $Spec$.

A C^∞-schemes and the functor Spec

Definition A.1 *1. A* **C^∞-ringed space** *$\underline{X} := (X, \mathcal{O}_X)$ is a topological space X with a sheaf $\mathcal{O}_X$ of C^∞-rings on X.*

2. *Let $\underline{X} = (X, \mathcal{O}_X), \underline{Y} = (Y, \mathcal{O}_Y)$ be C^∞-ringed spaces.*
   A morphism $\underline{f} = (f, f^\#) : \underline{X} \to \underline{Y}$ of C^∞-ringed spaces is a continuous map $f : X \to Y$ with a morphism $f^\# : f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$ of sheaves of C^∞-rings on X.
3. *Write* **C^∞RS** *for the category of C^∞-ringed space.*
4. *A* **local C^∞-ringed space** *$\underline{X} = (X, \mathcal{O}_X)$ is a C^∞-ringed space for which the limit $\mathcal{O}_{X,x} := \lim_{U \ni x} \mathcal{O}_X(U)$ for all open neighborhood of x is local for all $x \in X$.*
5. *Write* **LC$^\infty$RS** *for the full subcategory of* **C^∞RS** *of local C^∞-ringed space.*

Definition A.2 (The definition of functor Spec) *1. For a C^∞-ring $\mathfrak{C}$, define a C^∞-ringed space $\underline{X_\mathfrak{C}}$ as followings.*

   (a) Define a topological space $X_\mathfrak{C}$ as followings by C^∞-ring $\mathfrak{C}$.
   - *Define a set $X_\mathfrak{C} := \{x : \mathfrak{C} \to \mathbb{R} | x \text{ is a } \mathbb{R}\text{-point of } \mathfrak{C}.\}$.*
   - *For each $c \in \mathfrak{C}$, define $c_* : X_\mathfrak{C} \to \mathbb{R}$ as $c_*(x) := x(c)$.*
   - *Set a topology of $X_\mathfrak{C}$ as a smallest topology $\mathcal{T}_\mathfrak{C}$ such that c_* is continuous for all $c \in \mathfrak{C}$.*

   (b) For an open subset $U \subset X_\mathfrak{C}$, define $\mathcal{O}_{X_\mathfrak{C}}(U)$ as a set of functions $s : U \to \prod_{x \in U} \mathfrak{C}_x$ with following properties
   - *For each $x \in U$, $s(x) \in \mathfrak{C}_x$ is satisfied.*
   - *U is covered by open set V which satisfies that*
     there exists $c, d \in \mathfrak{C}(\forall x \in V, \pi_x(d) \neq 0)$ such that $\pi_x(c)\pi_x(d)^{-1} = s(x)(\forall x \in V)$.

2. *Therefore define the following C^∞-ringed space*

$$\text{Spec } \mathfrak{C} := (X_\mathfrak{C}, \mathcal{O}_{X_\mathfrak{C}}).$$

Definition A.3 (The definition of functor Spec) *1. For a morphism $\phi : \mathfrak{C} \to \mathfrak{D}$ of C^∞-rings, define $\underline{f_\phi} = (f_\phi, f_\phi^\#)$ as*

   (a) $f_\phi : X_\mathfrak{D} \ni x \mapsto (x \circ \phi) \in X_\mathfrak{C}$ is continuous,

   (b) $f_\phi^\# : \mathcal{O}_{X_\mathfrak{C}} \to f_\phi^*(\mathcal{O}_{X_\mathfrak{D}})$ is a morphism on $X_\mathfrak{D}$ which satsifies a following :
   for each open set $U \subset X_\mathfrak{C}$, define $f_\phi^\#(U) : \mathcal{O}_{X_\mathfrak{C}}(U) \to (f_\phi^*(\mathcal{O}_{X_\mathfrak{D}}))(U)$ as
   $f_\phi^\#(s) : x \mapsto \phi_x(s(f_\phi(x)))$ for each $s \in \mathcal{O}_{X_\mathfrak{C}}(U)$.

   (c) From the above, define a morphism as following

$$\text{Spec } \phi := (f_\phi, f_\phi^\#) : \underline{X_\mathfrak{D}} \to \underline{X_\mathfrak{C}}.$$

2. *From the above definitions of objects and morphisms, we can define a functor Spec of categories.*

$$\text{Spec} : \boldsymbol{C^\infty Rings}^{op} \to \boldsymbol{LC^\infty RS}$$

Definition A.4 *Let $\underline{X} = (X, \mathcal{O}_X)$ be a C^∞-ringed space.*

1. *A C^∞-ringed space $\underline{X}$ is called* **affine C^∞-scheme** *if it is isomorphic to Spec$\mathfrak{C}$ for some C^∞-ring $\mathfrak{C}$.*
2. *If there exists an open cover $\{\underline{U_\lambda}\}_{\lambda \in \Lambda}$ such that $\underline{U_\lambda}$ is an affine C^∞-scheme for each λ, we call $\underline{X}$* ***C^∞-scheme****.*
3. *We call C^∞-ringed space $\underline{X}$* **separated** *(resp. second countable, compact, paracompact) if the underlying topological space X is* **Hausdorff (resp. second countable, compact, paracompact)**.

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